18-848 Machine Learning for Radar

Wireless Basics II

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Modulation: make EM carry information

Guglielmo Marconi
1874-1937

Signal (in volt)

Carrier frequency

$$s(t) = A(t)\boxed{\cos(2\pi f_c t + \phi)}$$

"Carrier"

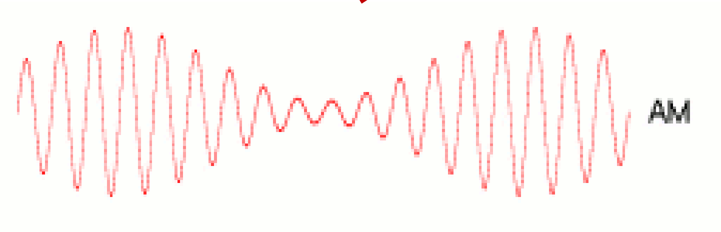
$$A(t) = \begin{cases} V, \text{key down} \\ 0, \text{key up} \end{cases}$$

Whether a key is tapped

The 0-1 code invented for this: Morse code.

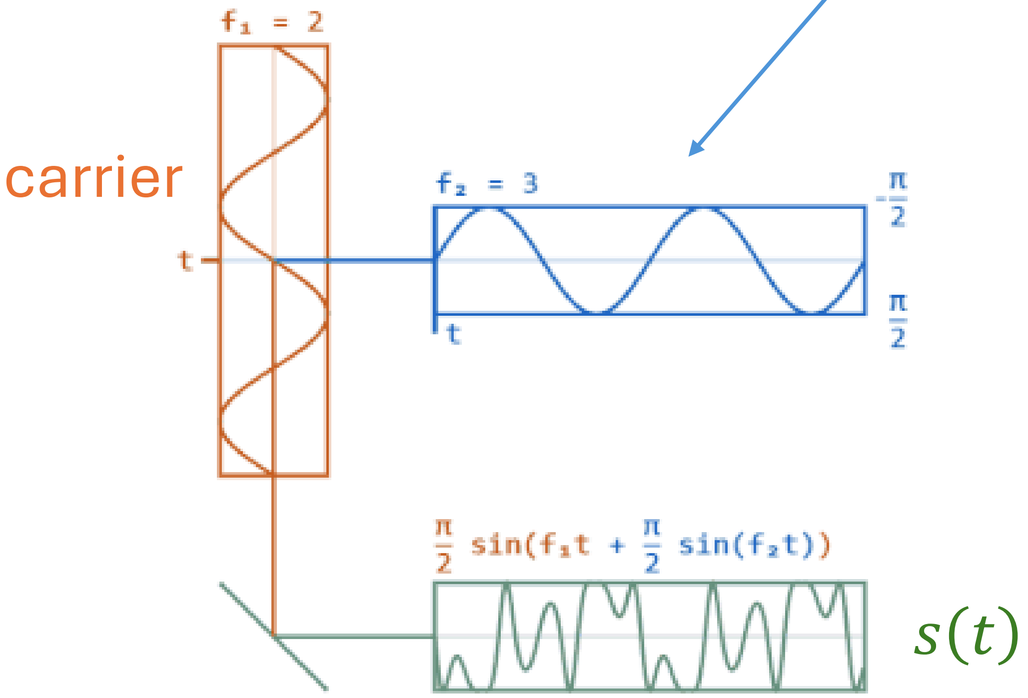
Amplitude Modulation

$$s(t) = A(t)\cos(2\pi f_c t + \phi)$$



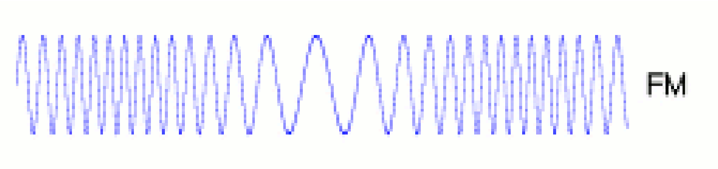
Phase Modulation

$$s(t) = A\cos\left(2\pi f_c t + \phi(t)\right)$$



Frequency Modulation

$$s(t) = A\cos 2\pi[f_c + f(t)]t + \phi$$



Complex representation

Consider a general, single-frequency signal:

$$s(t) = A\cos(2\pi f_c t + \phi)$$

Amplitude (Volt) **Phase** (Rad)

A "Phasor"

We can write it as:

$$s(t) = \Re\left[Ae^{j\phi}e^{j2\pi f_c t}\right] = \Re\left[x(t)e^{j2\pi f_c t}\right]$$

$$x(t) = Ae^{j\phi}$$ *Baseband signal*

Complex representation:
Make phase component **separable** from carrier.
From now on, we only consider the baseband $x(t)$.

Complex representation of modulations

Passband

$$s(t) = A(t)\cos(2\pi f_c t + \phi)$$

$$s(t) = A\cos\left(2\pi f_c t + \phi(t)\right)$$

$$s(t) = A\cos 2\pi[f_c + f(t)]t + \phi$$

These are the signals that are transmitted through the air.

Baseband

$$x(t) = A(t)e^{j\phi}$$

$$x(t) = Ae^{j\phi(t)}$$

$$x(t) = Ae^{j\left(2\pi\int f(\tau)\mathrm{d}\tau + \phi\right)}$$

These are the signals you see in your computer.

Practice: FMCW complex representation

$$s_{\mathrm{Tx}}(t) = A\cos(2\pi[f_c + f(t)]t + \phi)$$

$$f(t) = kt$$

$$s_{\mathrm{Rx}}(t) = A\cos(2\pi[f_c + f(t-\tau)](t-\tau) + \phi)$$

$$y(t) = s_{\mathrm{Tx}}(t) \cdot s_{\mathrm{Rx}}(t) = ?$$

FMCW complex representation

$$s_{\mathrm{Tx}}(t) = A\cos(2\pi[f_c + f(t)]t + \phi)$$

$$x_{\mathrm{Tx}}(t) = Ae^{j\left(\pi k t^2 + \phi\right)}$$

$$f(t) = kt$$

$$x_{\mathrm{Rx}}(t) = Ae^{j\left(\pi k (t-\tau)^2 + \phi - 2\pi f_c \tau\right)}$$

$$s_{\mathrm{Rx}}(t) = A\cos(2\pi[f_c + f(t-\tau)](t-\tau) + \phi)$$

$$\cancel{y(t) = s_{\mathrm{Tx}}(t) \cdot s_{\mathrm{Rx}}(t) =?}$$

$$x_{\mathrm{Tx}}(t) \cdot \overline{x_{\mathrm{Rx}}}(t) =?$$

FMCW complex representation

$$s_{\text{Tx}}(t) = A\cos(2\pi[f_c + f(t)]t + \phi)$$

$$x_{\text{Tx}}(t) = Ae^{j\left(\pi k t^2 + \phi\right)}$$

$$f(t) = kt$$

$$x_{\text{Rx}}(t) = Ae^{j\left(\pi k (t-\tau)^2 + \phi - 2\pi f_c \tau\right)}$$

$$s_{\text{Rx}}(t) = A\cos(2\pi[f_c + f(t-\tau)](t-\tau) + \phi)$$

$$\xcancel{y(t) = s_{\text{Tx}}(t) \cdot s_{\text{Rx}}(t) =?}$$

$$x_{\text{Tx}}(t) \cdot \overline{x_{\text{Rx}}}(t) = A^2 e^{j\left(2\pi k\tau t + 2\pi f_c \tau - \pi k \tau^2\right)}$$

FMCW complex representation

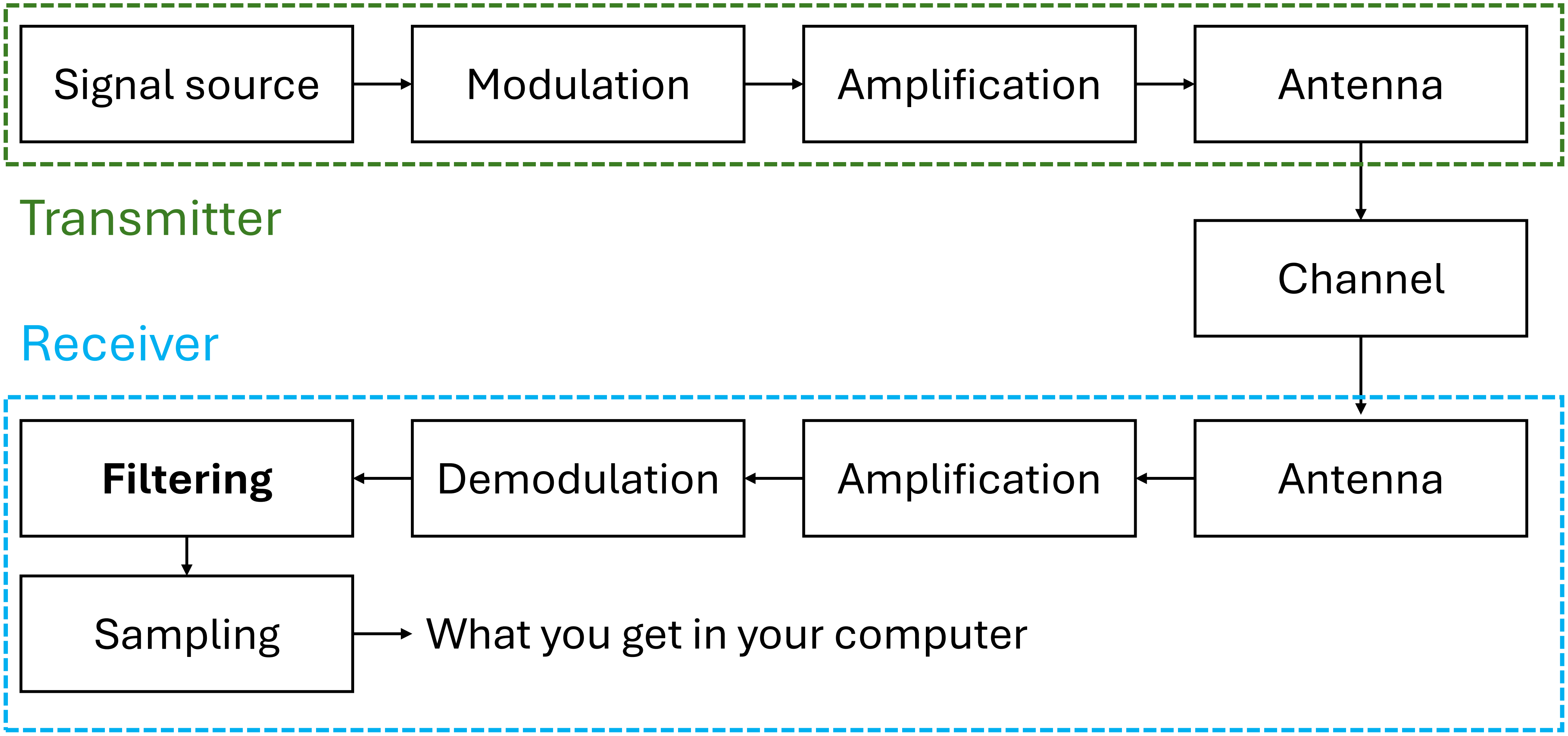
$$x(t) = x_{\mathrm{Tx}}(t) \cdot \overline{x_{\mathrm{Rx}}}(t) = A^2 e^{j\left(2\pi k\tau t + 2\pi f_c \tau - \pi k \tau^2\right)}$$

Frequency of $x(t)$: $k\tau$

"Beat frequency"

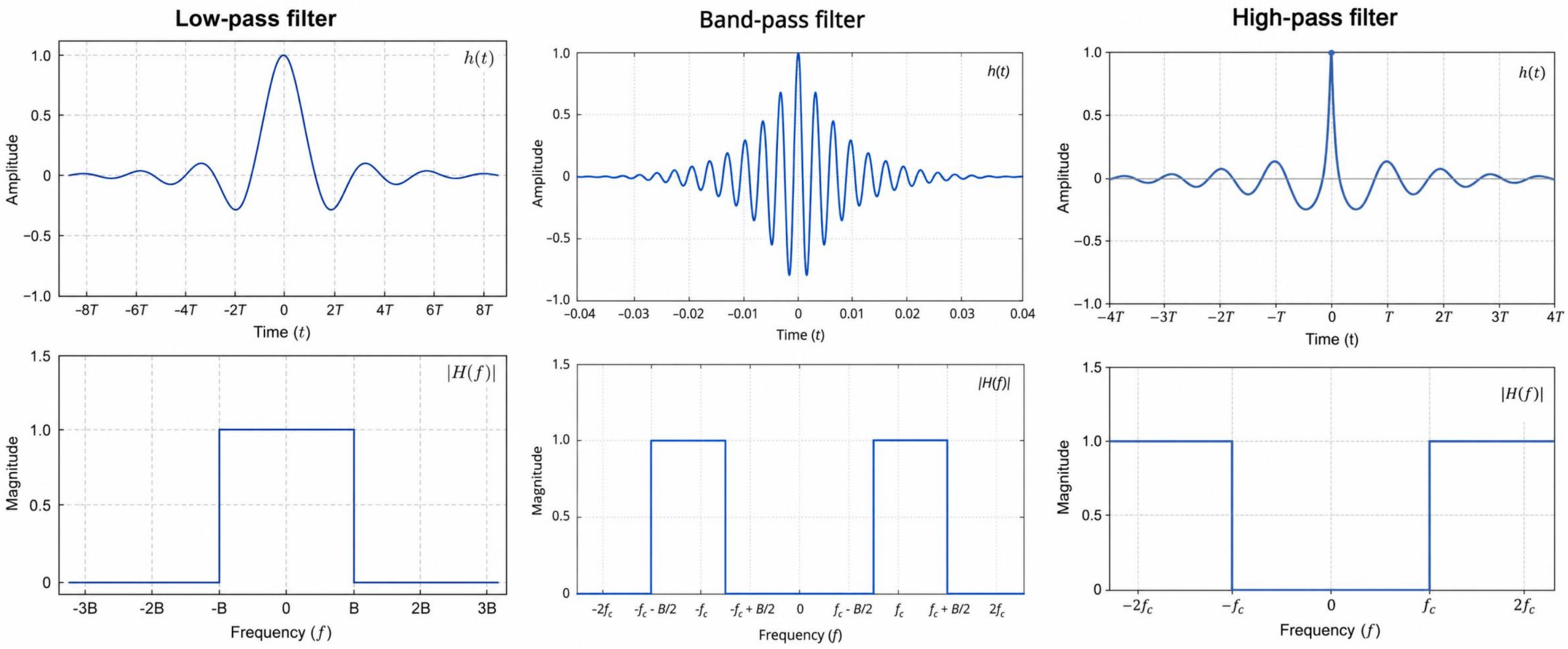
Phase of $x(t)$: $2\pi f_c \tau - \pi k \tau^2$

A wireless system diagram

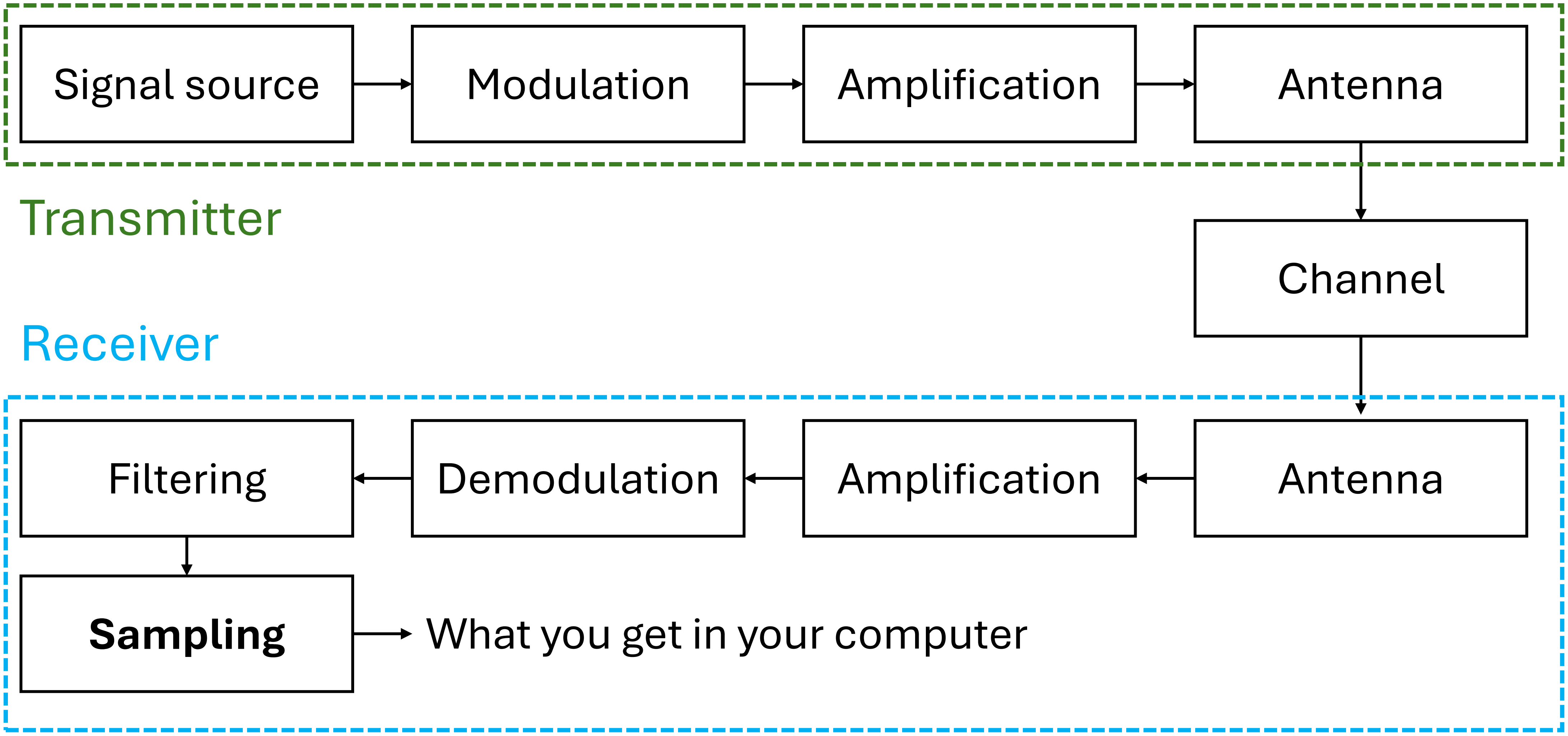


Filters

- Filters select frequency components of specific ranges.



A wireless system diagram



Sampling

- Our computer only computes discrete values.
- How to analyze continuous signal?
- We **sample** it. Define sample interval T_S and sample frequency f_S

$$f_S = 1/T_S$$

- We have the sampled signal

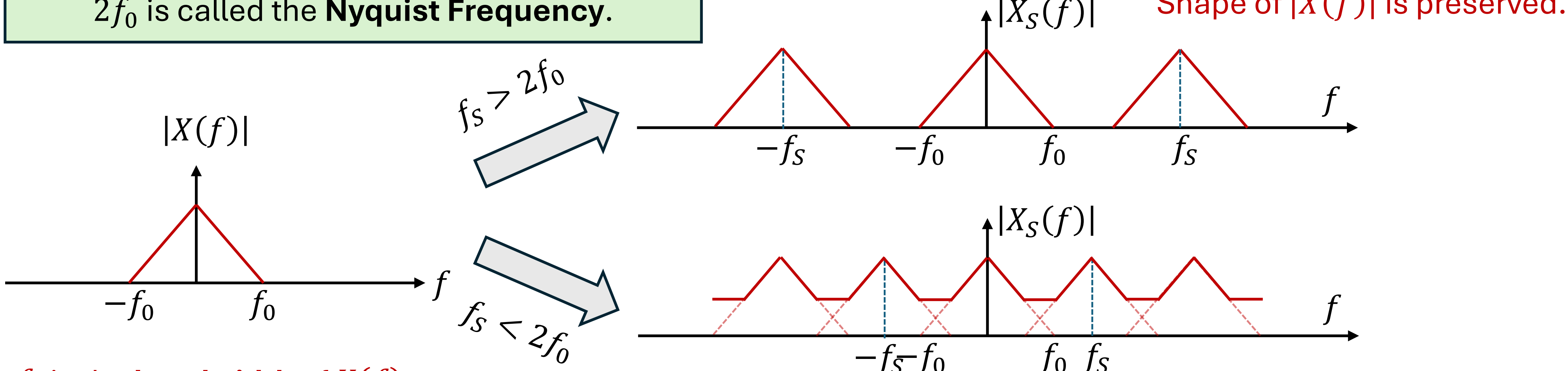
$$x[n] = x(nT_S) = x\left(\frac{n}{f_S}\right), n \in \mathbb{Z}$$

https://www.youtube.com/watch?v=eBHbCZo9QrM&pp=0gcJCRoMAYcqIYzv

Nyquist rate and aliasing

$$x(nT_s) \leftrightarrow X_S(f) = \frac{1}{T_S} \sum_{k=-\infty}^{\infty} X(f - k f_S)$$

Nyquist Sampling Theorem
To prevent aliasing, we must have $f_S > 2f_0$.
$2f_0$ is called the **Nyquist Frequency**.



f_0 is the **bandwidth** of $X(f)$.

Shape of $|X(f)|$ is distorted: **Aliasing.**

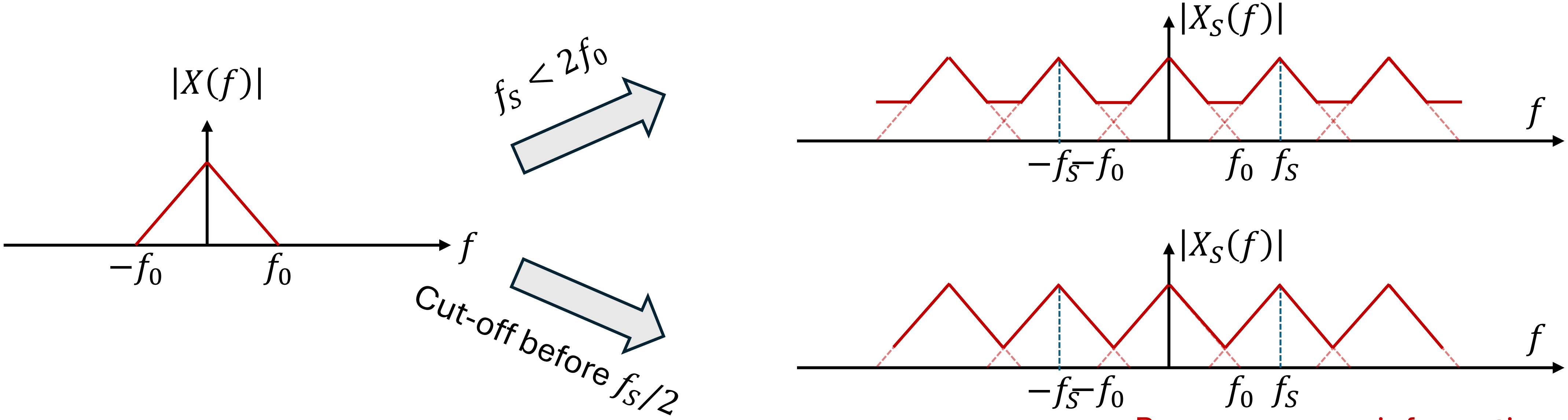
A comparison table (in the radio spectrum)

Property	Lower frequencies	Higher frequencies
Propagation distance	Larger	Smaller
Ability to penetrate through occlusions	Strong	Weak
Behaves like	“Wave”	“Particle” / “Beam”
Directivity	Usually omni-directional	Usually directional
Bandwidth	**Small (Carries less information)**	**Large (Carries more information)**

Anti-aliasing filter

In practical cases, we usually have signals that operate across a very wide of frequencies.
But to reduce computing, we have to use a low sampling rate.

We can remove the signal with frequency higher than $f_S/2$: **anti-aliasing filter**.



Preserves more information.

Discrete-Time Fourier Transform (DTFT)

For CTFT, $$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \mathrm{d}t$$

For DTFT, $$X(f) = \sum_{-\infty}^{\infty} x[n] e^{-j2\pi f n T_S}$$

There is $X(f \pm f_s) = X(f)$ Why?

Discrete-Time Fourier Transform (DTFT)

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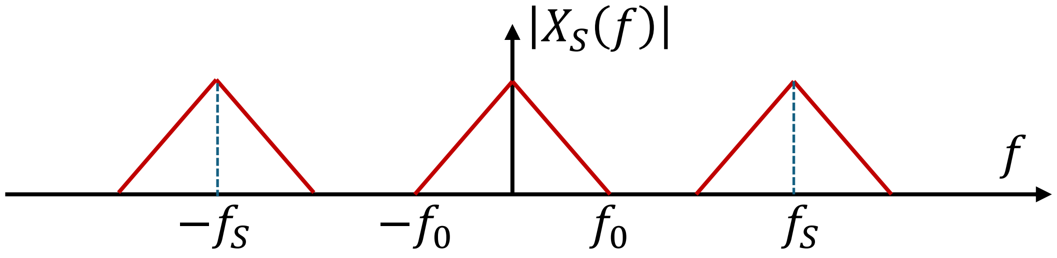
There is $$X(f \pm f_s) = X(f)$$ Why?

Because $x[n]$ is sampled from $x(t)$ at f_S.



Discrete-Time Fourier Transform (DTFT)

For CTFT, $$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\mathrm{d}t$$

For DTFT, $$X(f) = \sum_{-\infty}^{\infty} x[n]e^{-j2\pi fnT_S}$$

There is $$X(f \pm f_s) = X(f)$$

So we usually only consider $-\frac{f_S}{2} \leq f < \frac{f_S}{2}$.

Discrete Fourier Transform (DFT)

- But wait, frequency domain for DTFT is still continuous.
- How do we process frequency domain in the computer?
- By making the frequency domain also discrete.

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi kn}{N}}$$

Use **fftshift** function to shift result to negative-0-positive structure

Note: $k = 0, 1, \cdots, N-1$ correspond to

$$f = \underbrace{0, \frac{f_S}{N}, \cdots, \left(\frac{\lfloor N/2 \rfloor - 1}{N}\right) f_S}_{\text{Positive frequency}}, \underbrace{-\left(1 - \frac{\lceil N/2 \rceil}{N}\right) f_S, \cdots, -\frac{f_S}{N}}_{\text{Negative frequency}}$$

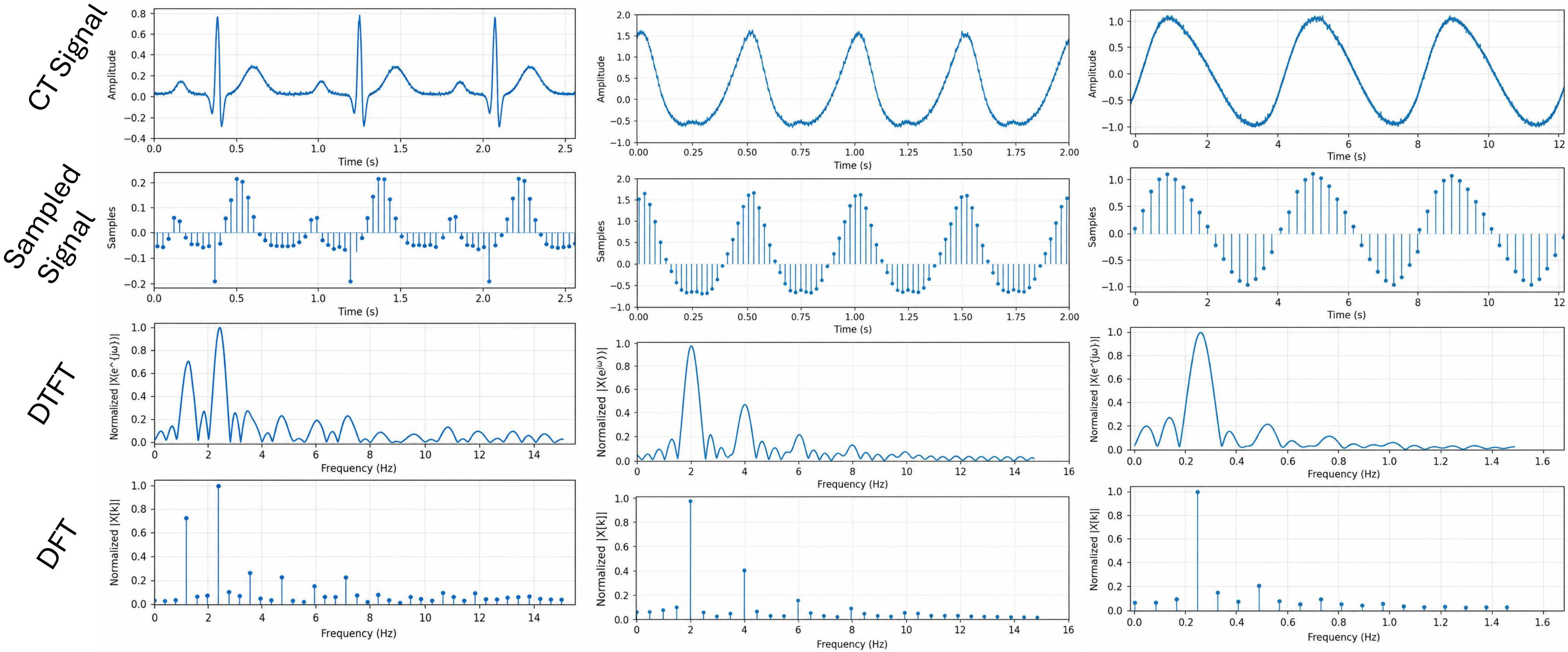
A comparison between CTFT, DTFT, DFT

Representation	Time-domain	Frequency-domain
CTFT	Continuous	Continuous
DTFT	Discrete	Continuous
DFT	Discrete	Discrete
FS	Continuous	Discrete

Fast Fourier Transform (FFT)

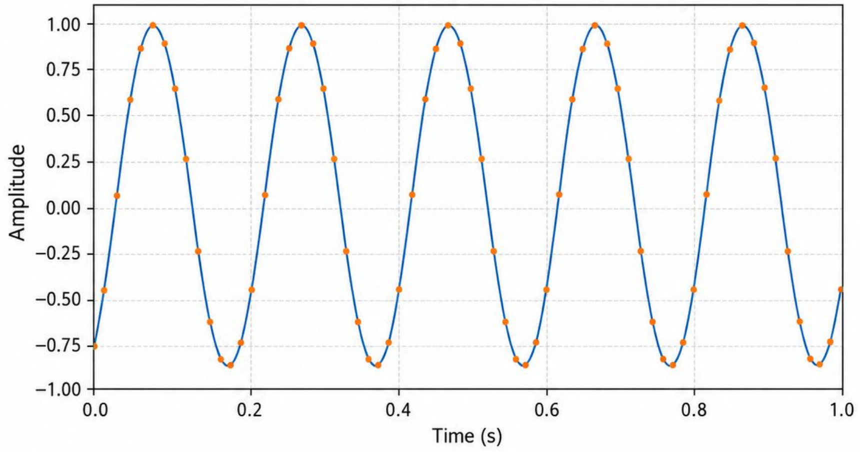
- A fast way to calculate DFT.
- A classic divide-and-conquer algorithm.
- Learn it if you are interested in related jobs.

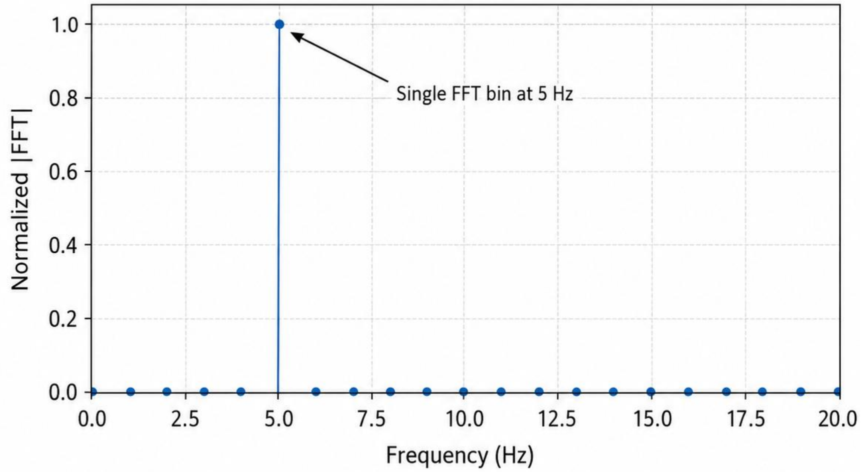
Examples of DTFT and DFT



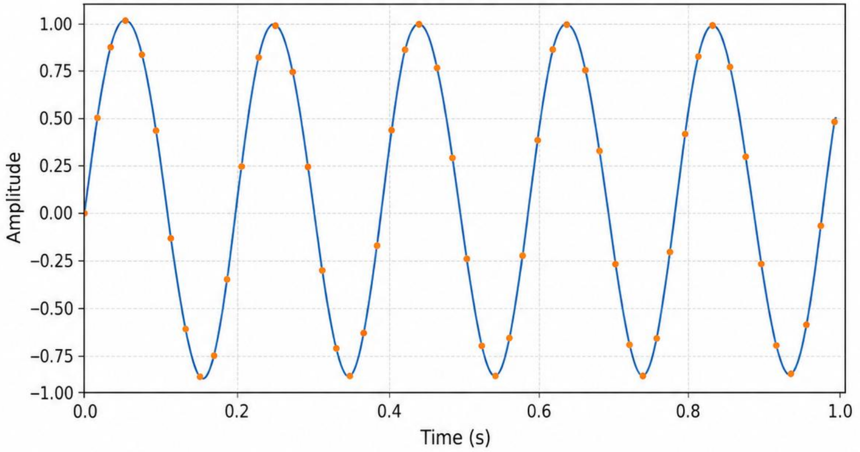
Spectral leakage of FFT

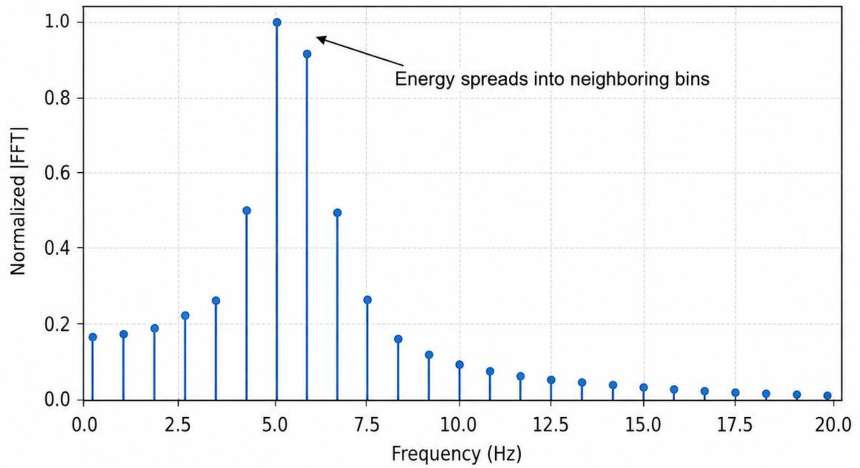
When a sine wave is sampled exactly with $kf_S = Nf_0$, the FFT result is perfect:





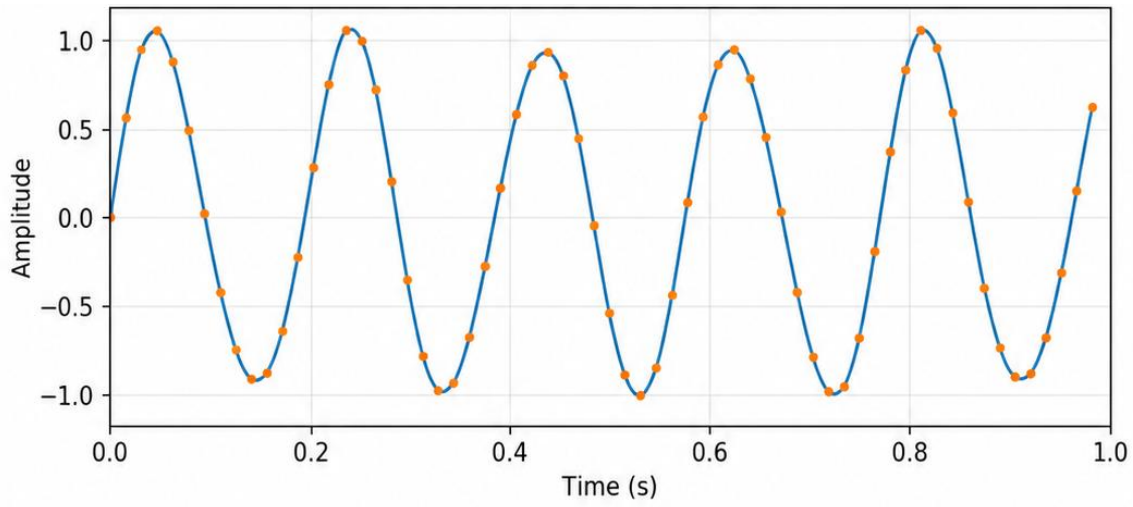
However, when $kf_S \neq Nf_0$ and the wave is cut sharply, there is **spectral leakage**:

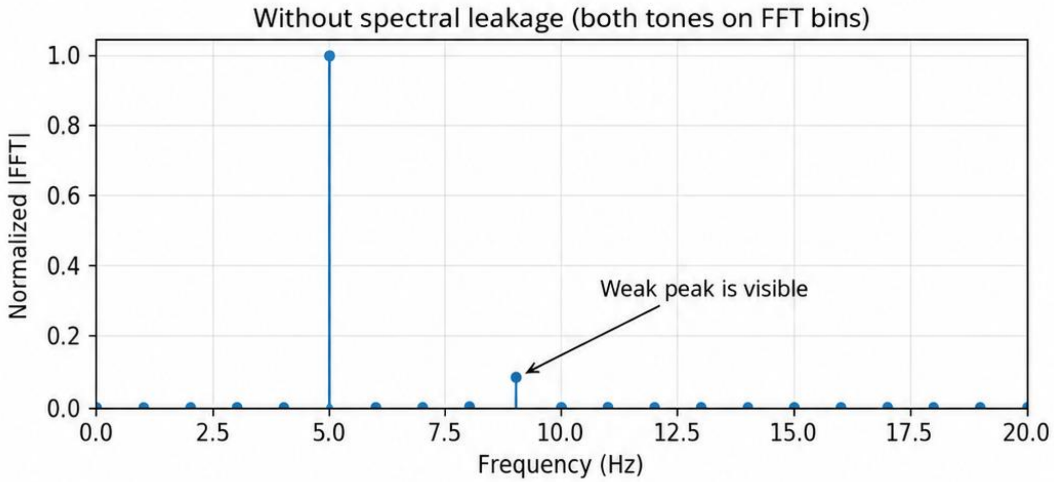


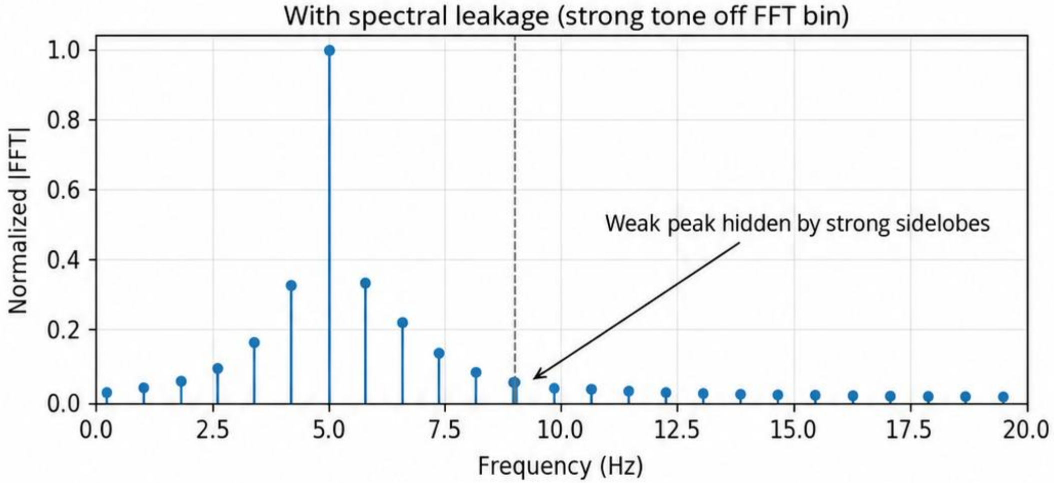


Spectral leakage covers weak objects

If we have a combination of two sine waves,
One strong and one weak:





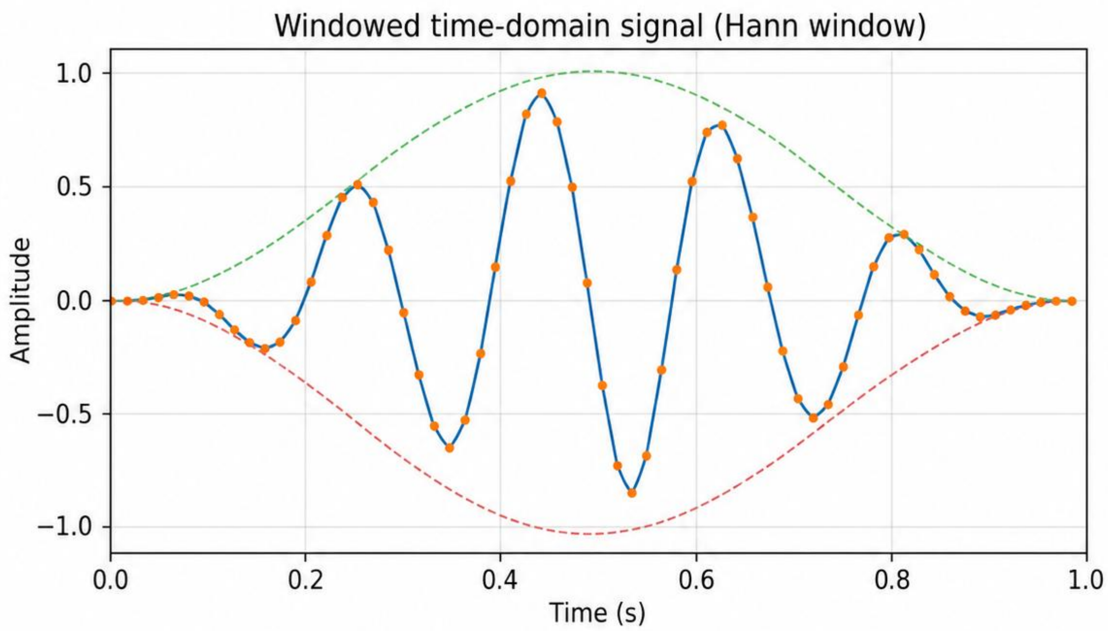


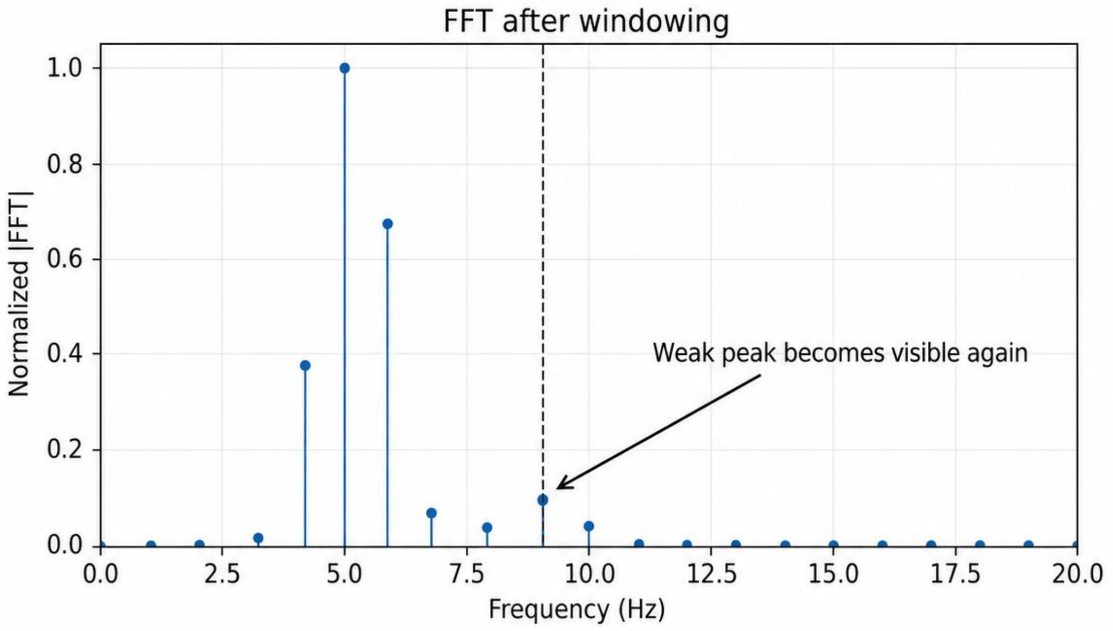
Windowing

- Make the signal edge decrease gradually to 0.

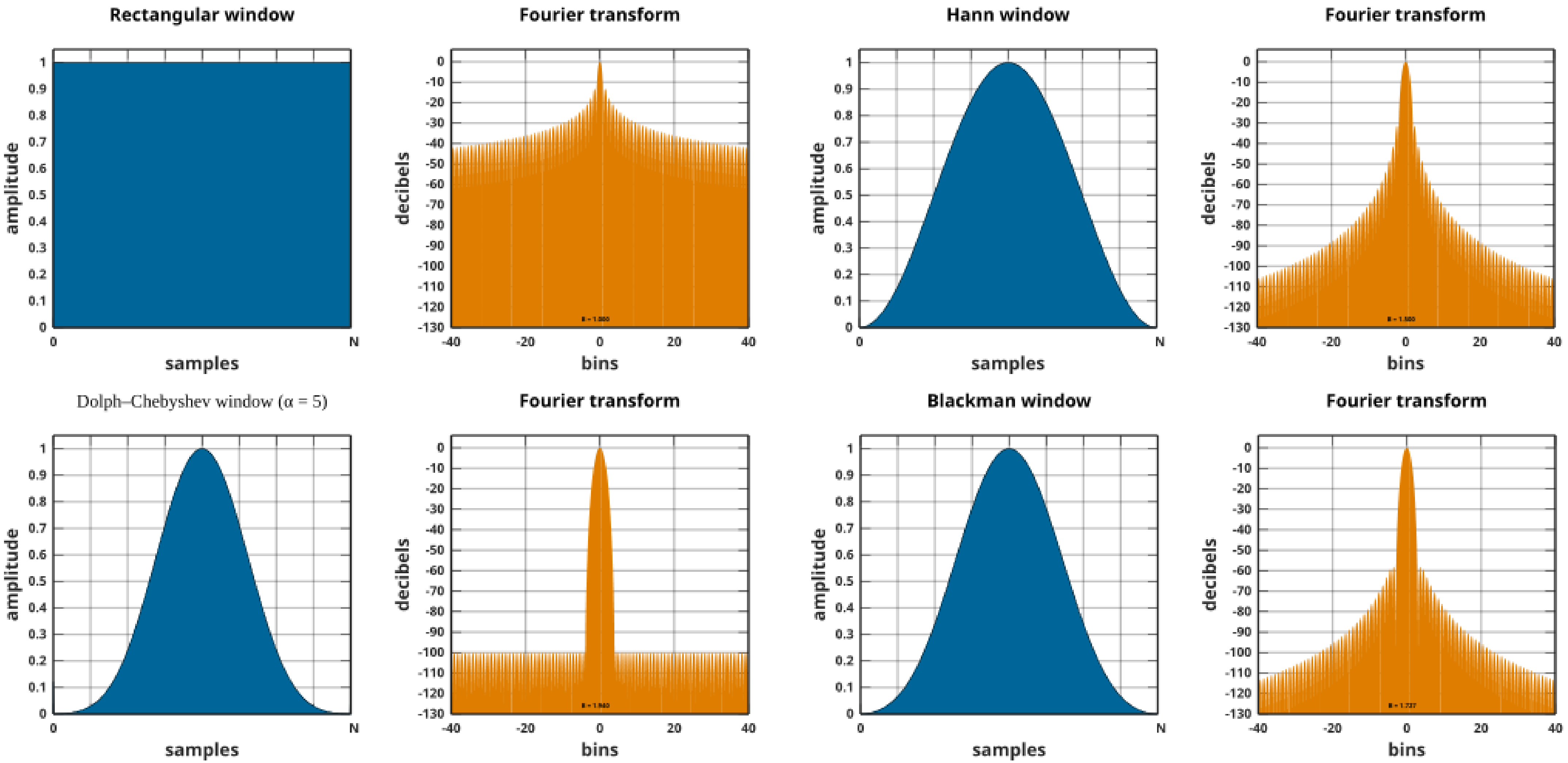
$$x_W[n] = w[n] \cdot x[n]$$

- $w[n]$ is the **window**.





Example windows (Wikipedia: window function)

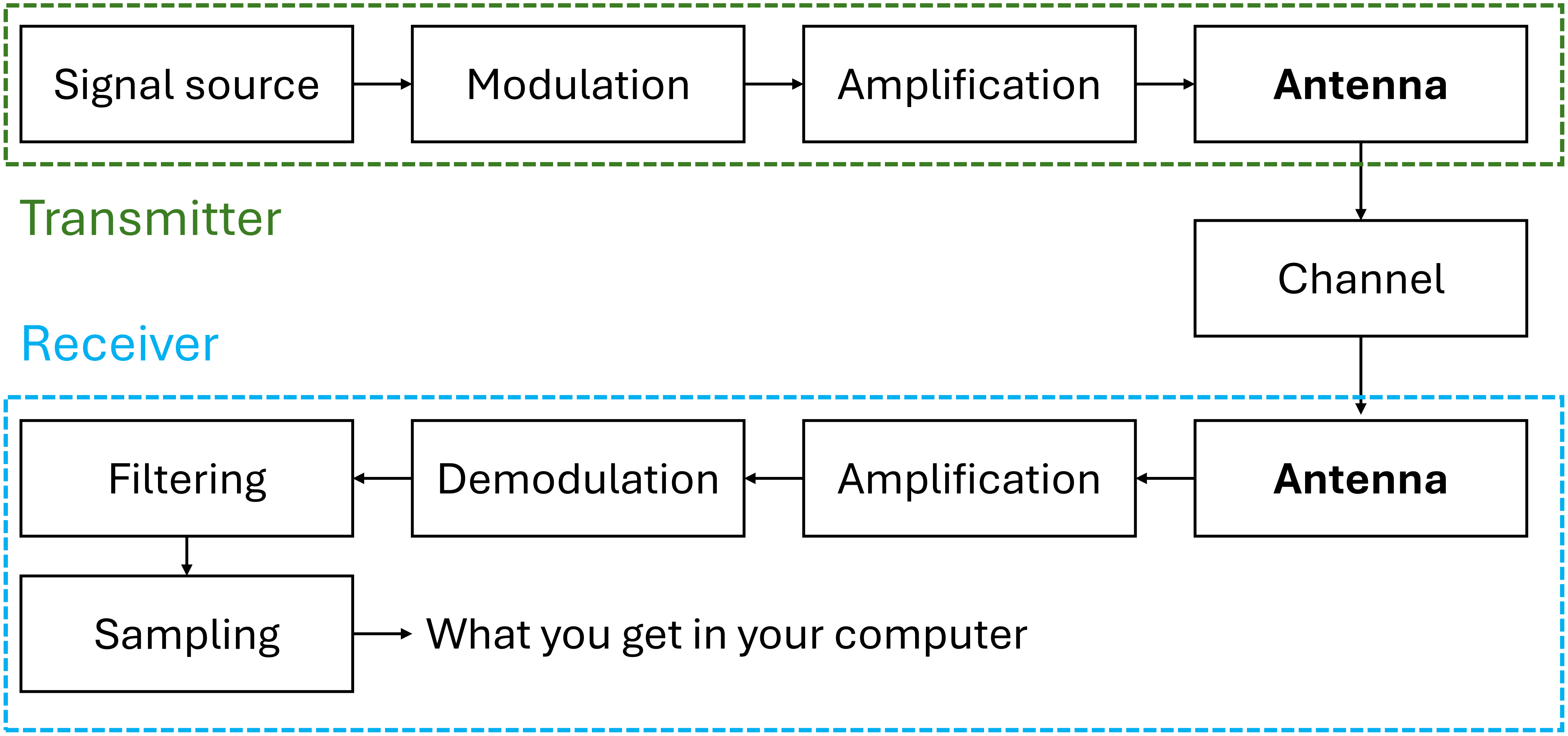


A comparison between common windows

Window	Resolution	Sidelobe suppression
Rectangular	Best	Poor
Hann	Good	Good
Blackman-Harris	Worse	Excellent
Taylor	Tunable	Tunable

Select windows according to the scenario your are dealing with.

A wireless system diagram

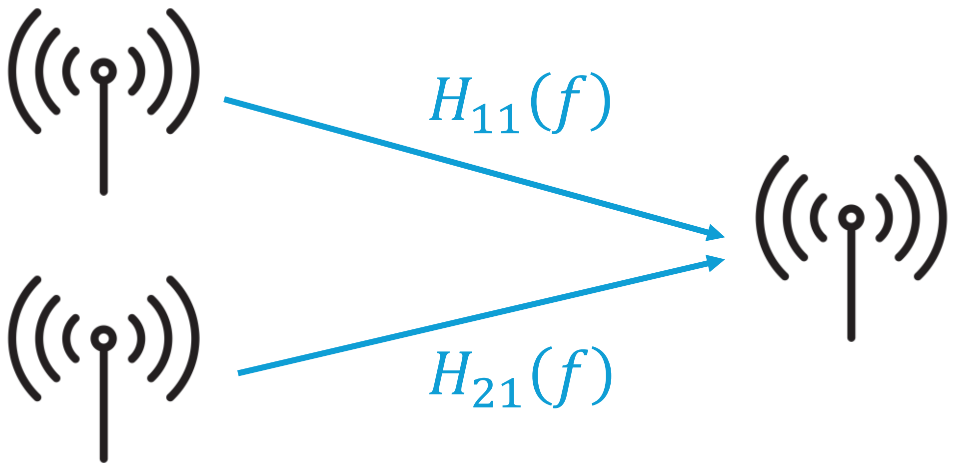


Multi-antenna systems

"SISO"

Recap: A **Single-Input-Single-Output** system

$$Y(f) = H(f)X(f)$$



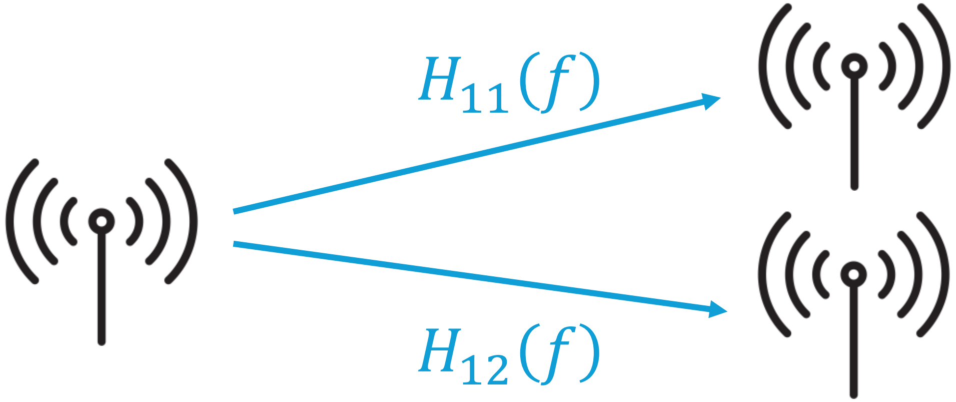
"MISO"

Extend to **Multi-Input-Single-Output** system

$$\begin{aligned} Y(f) &= H_{11}(f)X_1(f) + H_{21}(f)X_2(f) \\ &= \begin{bmatrix} H_{11}(f) & H_{21}(f) \end{bmatrix} \begin{bmatrix} X_1(f) \\ X_2(f) \end{bmatrix} \end{aligned}$$

$$Y(f) = \boldsymbol{H}(f)\boldsymbol{X}(f)$$

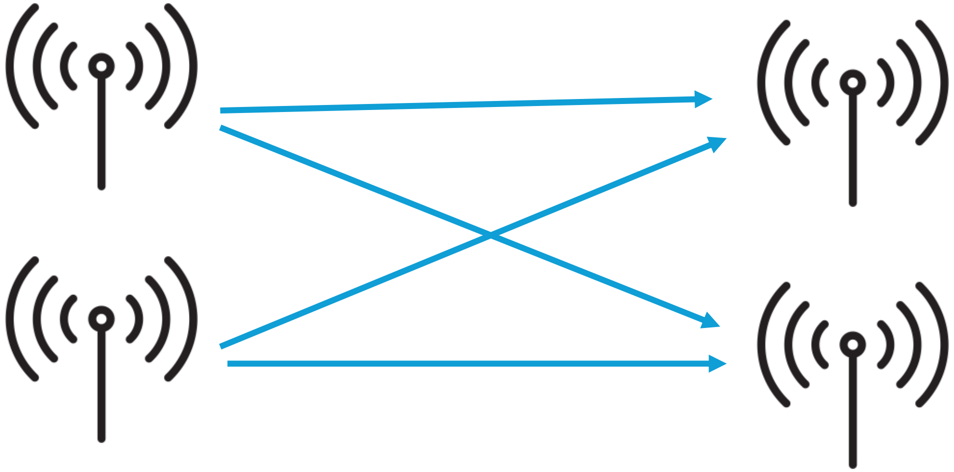
Multi-antenna systems



"SIMO"

Single-Input-Multi-Output system

$$Y_1(f) = H_{11}(f)X(f)$$
$$Y_2(f) = H_{12}(f)X(f)$$
$$\boldsymbol{Y}(f) = \boldsymbol{H}(f)X(f)$$



"MIMO"

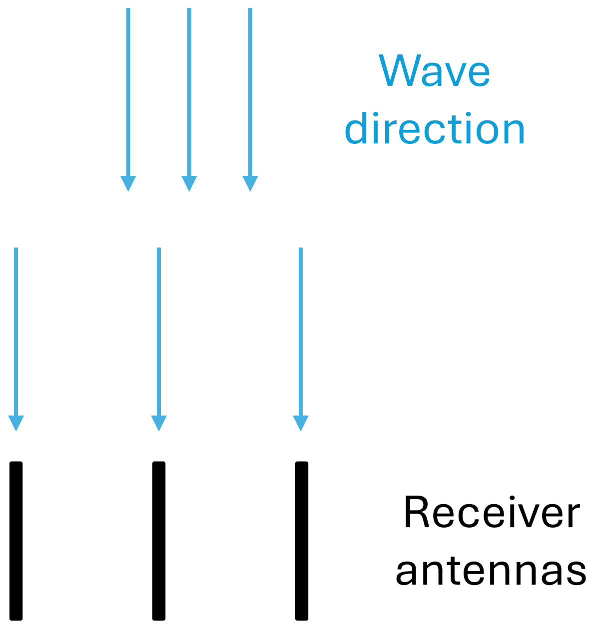
Multi-Input-Multi-Output system

$$\begin{bmatrix} Y_1(f) \\ Y_2(f) \end{bmatrix} = \begin{bmatrix} H_{11} & H_{21} \\ H_{12} & H_{22} \end{bmatrix} \begin{bmatrix} X_1(f) \\ X_2(f) \end{bmatrix}$$

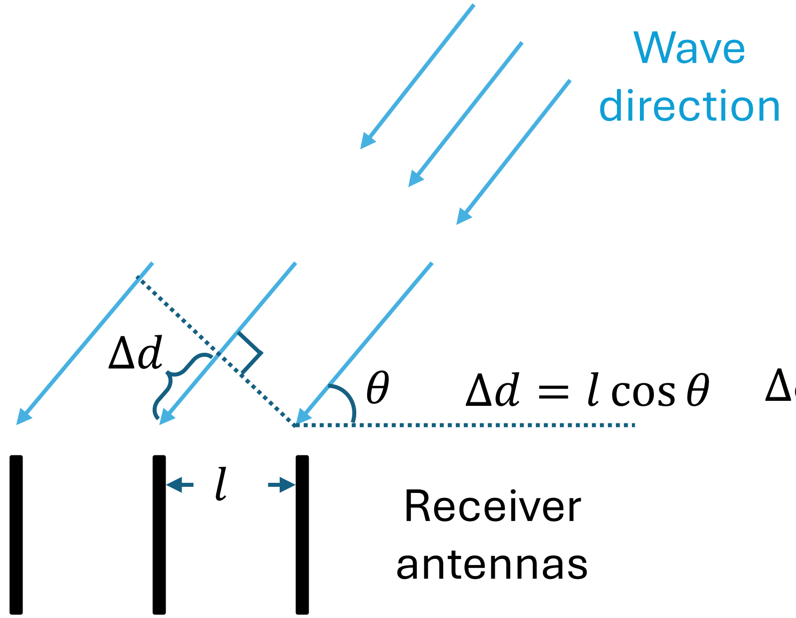
$$\boldsymbol{Y}(f) = \boldsymbol{H}(f)\boldsymbol{X}(f)$$

Angle-of-arrival (AoA)

- An important benefit given by multi-antenna systems is to do AoA estimation.



$$H_3(f) = H_2(f) = H_1(f)$$



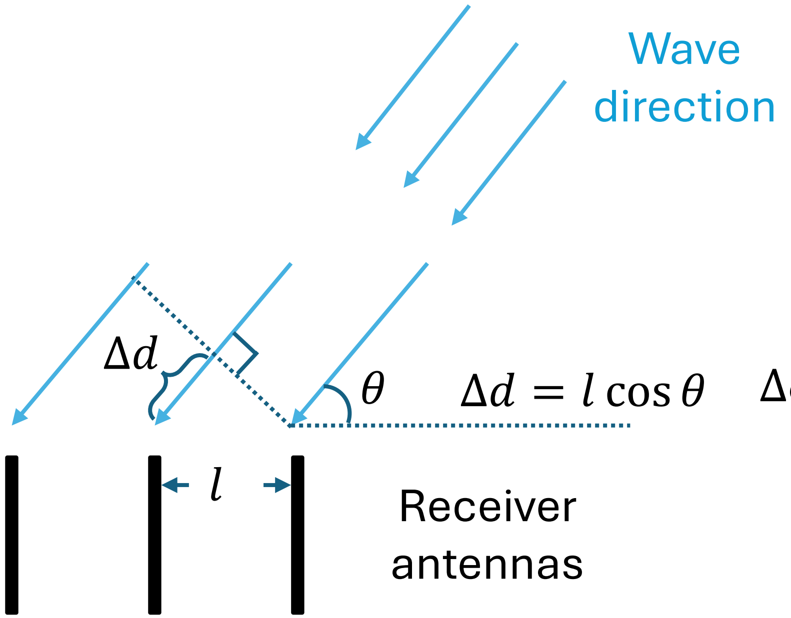
$$\Delta d = l \cos\theta \qquad \Delta\phi = \frac{2\pi l \cos\theta}{\lambda}$$

$$H_1(f)$$

$$H_2(f) = H_1(f) \cdot e^{j2\pi\frac{l\cos\theta}{\lambda}}$$

$$H_3(f) = H_1(f) \cdot e^{j4\pi\frac{l\cos\theta}{\lambda}}$$

Angle-of-arrival (AoA)



$$\Delta d = l\cos\theta \qquad \Delta\phi = \frac{2\pi l\cos\theta}{\lambda}$$

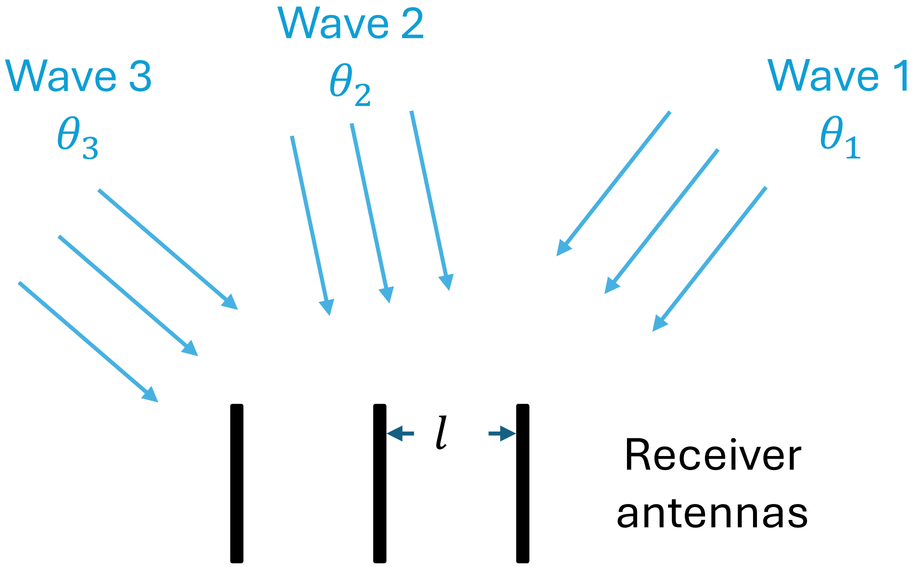
$$H_n(f) = H_1(f)\cdot e^{j(n-1)2\pi\frac{l\cos\theta}{\lambda}}$$

Steering vector

$$\boldsymbol{H}(f) = H_1(f)\cdot \boxed{\begin{bmatrix} 1 \\ \vdots \\ e^{j(n-1)2\pi\frac{l\cos\theta}{\lambda}} \end{bmatrix}}$$

Angle-of-arrival (AoA)

$$\boldsymbol{H}(f) = \underbrace{\boxed{H_{1,1}(f) \cdot \begin{bmatrix} 1 \\ \vdots \\ e^{j(n-1)2\pi\frac{l\cos\theta_1}{\lambda}} \end{bmatrix}}}_{\text{Wave 1}} + \underbrace{\boxed{H_{1,2}(f) \cdot \begin{bmatrix} 1 \\ \vdots \\ e^{j(n-1)2\pi\frac{l\cos\theta_2}{\lambda}} \end{bmatrix}}}_{\text{Wave 2}} + \underbrace{\boxed{H_{1,3}(f) \cdot \begin{bmatrix} 1 \\ \vdots \\ e^{j(n-1)2\pi\frac{l\cos\theta_3}{\lambda}} \end{bmatrix}}}_{\text{Wave 3}}$$



$$H_n(f) = H_1(f) \cdot e^{j(n-1)2\pi\frac{l\cos\theta}{\lambda}}$$

More generally,

$$\boldsymbol{H}(f) = \sum_{k=1}^{K} H_{1,k}(f) \begin{bmatrix} 1 \\ \vdots \\ e^{j(n-1)2\pi\frac{l\cos\theta_k}{\lambda}} \end{bmatrix}$$

k-th multipath

Complex gain of the k-th multipath

AoA Estimation

Solve θ_k from $\boldsymbol{H}(f) = \sum_{k=1}^{K} H_{1,k}(f) \begin{bmatrix} 1 \\ \vdots \\ e^{j(n-1)2\pi \frac{l \cos \theta_k}{\lambda}} \end{bmatrix}$

You will learn this from Radar Basics Lectures.

Question

- How many AoAs can we possibly distinguish given N antennas?

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- How many AoAs can we possibly distinguish given N antennas?

K angles and complex path gains

$$\boldsymbol{H}(f) = \sum_{k=1}^{K} H_{1,k}(f) \begin{bmatrix} 1 \\ \vdots \\ e^{j(n-1)2\pi\frac{l\cos\theta_k}{\lambda}} \end{bmatrix}$$

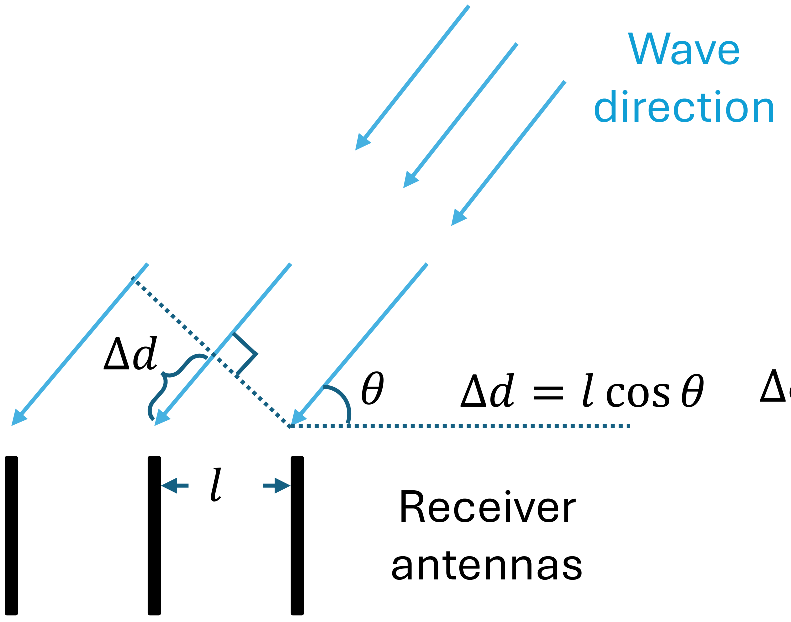
N complex measurements

Ideally, $K \leq N - 1$

In practice: Resolution is often the limit.

Beamforming

- We can control the **phase difference** of the transmitting antenna to create **directional waves (beams)**.

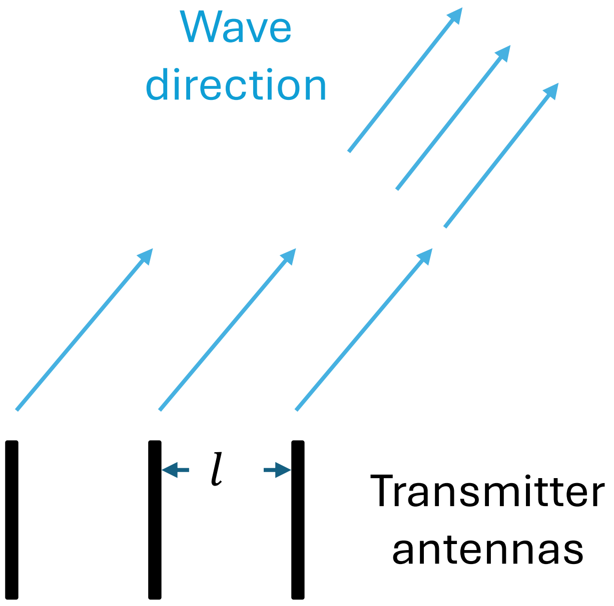


$$\Delta\phi = \frac{2\pi l \cos\theta}{\lambda}$$

$$H_1(f)$$

$$H_2(f) = H_1(f) \cdot e^{j2\pi\frac{l\cos\theta}{\lambda}}$$

$$H_3(f) = H_1(f) \cdot e^{j4\pi\frac{l\cos\theta}{\lambda}}$$

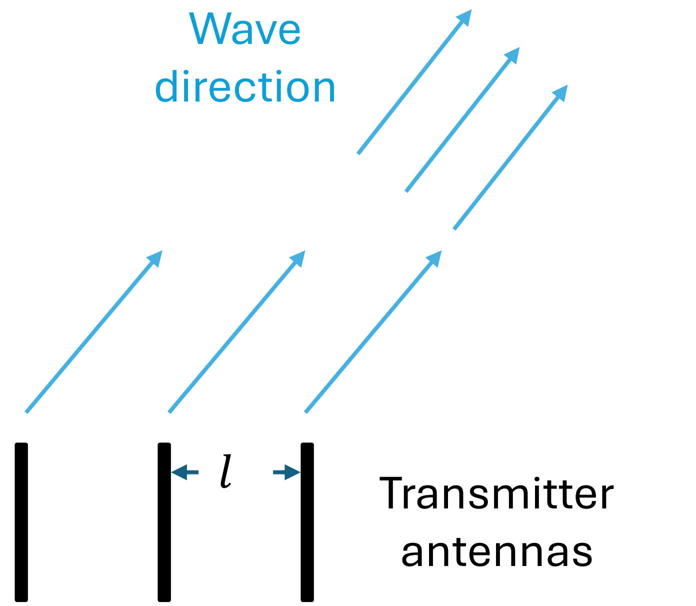


$$X_1(f)$$

$$X_2(f) = X_1(f) \cdot e^{-j2\pi\frac{l\cos\theta}{\lambda}}$$

$$X_3(f) = X_1(f) \cdot e^{-j4\pi\frac{l\cos\theta}{\lambda}}$$

Beamforming



$$X_n(f) = X_1(f) \cdot e^{-j(n-1)2\pi\frac{l\cos\theta}{\lambda}}$$

Steering vector

$$\boldsymbol{X}(f) = X_1(f) \cdot \boxed{\begin{bmatrix} 1 \\ \vdots \\ e^{-j(n-1)2\pi\frac{l\cos\theta}{\lambda}} \end{bmatrix}}$$

Comparison: AoA and Beamforming

	AoA	Beamforming
Antenna array	Receiver	Transmitter
Steering vector	$\begin{bmatrix} 1 \\ \vdots \\ e^{j(n-1)2\pi\frac{l\cos\theta}{\lambda}} \end{bmatrix}$	$\begin{bmatrix} 1 \\ \vdots \\ e^{-j(n-1)2\pi\frac{l\cos\theta}{\lambda}} \end{bmatrix}$
Steering angles	Multiple	Usually 1
Operates on	Channel H	Transmitted signal X